Fundamentals of Accelerators
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Day 2

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The Lorentz transformation

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \quad , \quad t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}$$

$$y' = y \quad , \quad z' = z$$

Or in matrix form

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z' \end{pmatrix}$$

Proper time & length

- We define the proper time, τ, as the duration measured in the rest frame

- The length of an object in its rest frame is L_o

- As seen by an observer moving at v, the duration, $\mathscr{T}$, is

$$\mathscr{T} = \frac{\tau}{\sqrt{1 - v^2/c^2}} \equiv \gamma\tau > \tau$$

And the length, L, is

$$L = L_o/\gamma$$

Four-vectors

- Introduce 4-vectors, w^α, with 1 time-like and 3 space-like components (α = 0, 1, 2, 3)
  - $x^\alpha = (ct, x, y, z)$ [Also, $x_\alpha = (ct, -x, -y, -z)$
  - Note Latin indices i =1, 2, 3
- Norm of w^α is $$|w| = (w^\alpha w_\alpha)^{1/2} = (w_o^2 - w_1^2 - w_2^2 - w_3^2)^{1/2}$$

$|w|^2 = g_{\mu\nu} w^\mu w^\nu$ where the metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Velocity, energy and momentum

- For a particle with 3-velocity ***v***, the 4-velocity is

$$u^{\alpha} = (\gamma c, \gamma \mathbf{v}) = \frac{dx^{\alpha}}{d\tau}$$

- The total energy, E, of a particle is its rest mass, m_o, plus kinetic energy, T

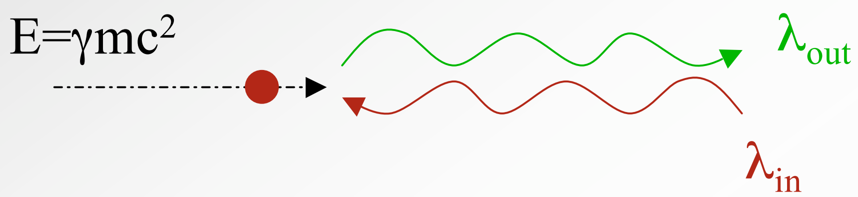
$$E = m_o c^2 + T = \gamma m_o c^2$$

- The 4-momentum, p^{μ}, is

$$p^{\mu} = (c\gamma m_0, \gamma m_0 \mathbf{v})$$

$$p^2 = m_o^{\ 2} c^2$$

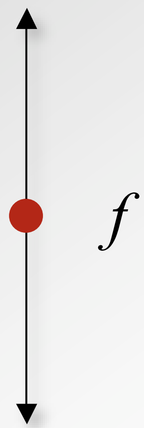
Head-on Compton scattering by an ultra-relativistic electron

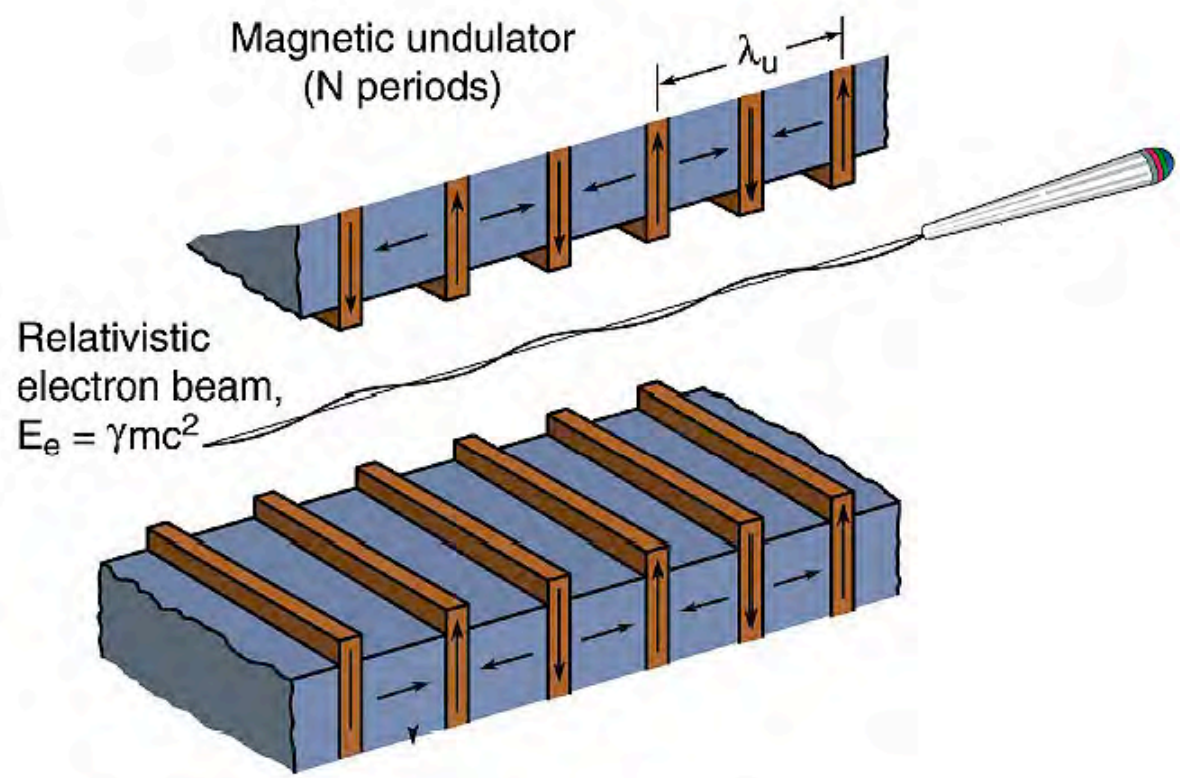


- What wavelength is the photon scattered by 180°?

Undulator radiation: What is λ_{rad}?

An electron in the lab oscillating at frequency, f, emits dipole radiation of frequency f

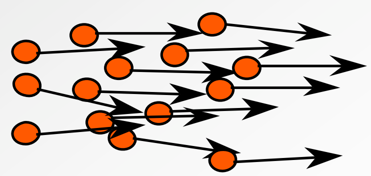




What about the relativistic electron?

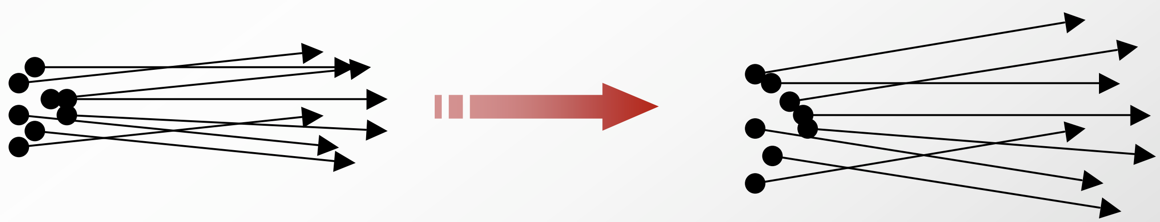
Some other characteristics of beams

- Beams particles have random (thermal) ⊥ motion



$$\vartheta_x = \left\langle \frac{p_x^x}{p_z^2} \right\rangle^{1/2} > 0$$

- Beams must be confined against thermal expansion during transport



Beams have internal (self-forces)

- Space charge forces
  - Like charges repel
  - Like currents attract
- For a long thin beam

$$E_{sp}(V/cm) = \frac{60\, I_{beam}(A)}{R_{beam}(cm)}$$

$$B_{\theta}(gauss) = \frac{I_{beam}(A)}{5\, R_{beam}(cm)}$$

Net force due to transverse self-fields

In vacuum:

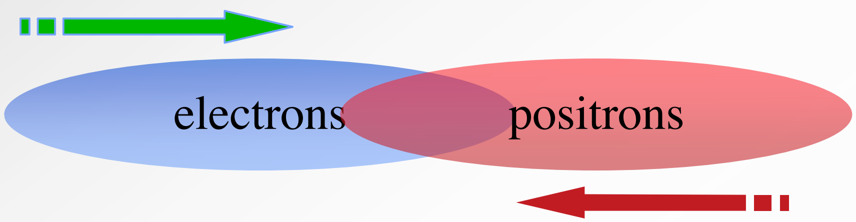
Beam's transverse self-force scale as $1/\gamma^2$

- Space charge repulsion: $E_{sp,\perp} \sim N_{beam}$
- Pinch field: $B_\theta \sim I_{beam} \sim v_z N_{beam} \sim v_z E_{sp}$

$$\therefore F_{sp,\perp} = q (E_{sp,\perp} + v_z \times B_\theta) \sim (1-v^2) N_{beam} \sim N_{beam}/\gamma^2$$

Beams in collision are ***not*** in vacuum (beam-beam effects)

Example: Megagauss fields in linear collider



At Interaction Point space charge cancels; currents add

==> strong beam-beam focus

--> Luminosity enhancement

--> Strong synchrotron radiation

Consider 250 GeV beams with 1 kA focused to 100 nm

$$B_{peak} \sim 40 \text{ Mgauss}$$

The Basics - Mechanics

Newton's law

- We all know

$$\mathbf{F} = \frac{d}{dt}\mathbf{p}$$

- The 4-vector form is

$$F^{\mu} = \left(\gamma c \frac{dm}{dt}, \gamma \frac{d\mathbf{p}}{dt}\right) = \frac{dp^{\mu}}{d\tau}$$

- Differentiate $p^2 = m_o{}^2 c^2$ with respect to τ

$$p_{\mu}\frac{dp^{\mu}}{d\tau} = p_{\mu}F^{\mu} = \frac{d(mc^2)}{dt} - \mathbf{F} \circ \mathbf{v} = 0$$

- The work is the rate of changing mc^2

Harmonic oscillator

- Motion in the presence of a linear restoring force

$$F = -kx$$

$$\ddot{x} + \frac{k}{m} x = 0$$

$$x = A \ \sin \omega_o t \ \text{ where } \ \omega_o = \sqrt{k/m}$$

- It is worth noting that the simple harmonic oscillator is a linearized example of the pendulum equation

$$\ddot{x} + \omega_o^2 \sin(x) \approx \ddot{x} + \omega_o^2 (x - x^3/6) = 0$$

that governs free electron laser instability

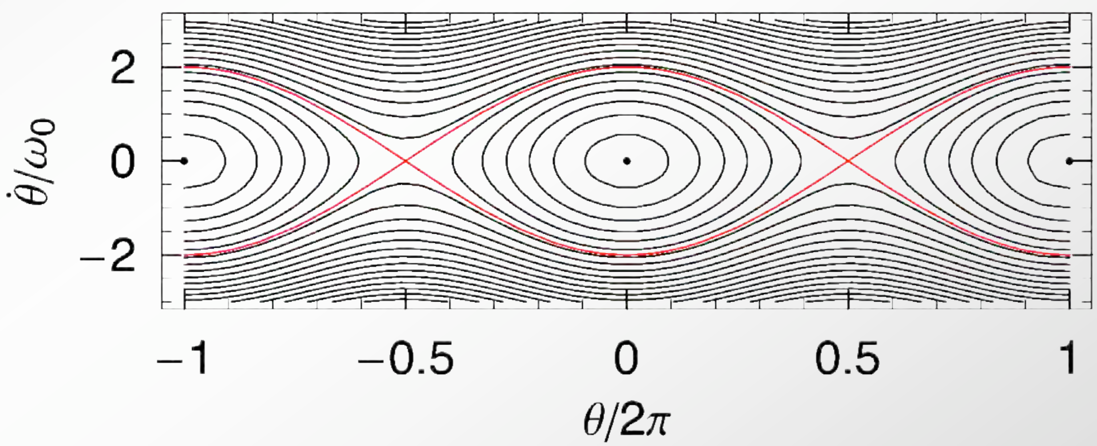
Solution to the pendulum equation

- Use energy conservation to solve the equation exactly
- Multiply $\ddot{x} + \omega_o^2 \sin(x) = 0$ by $\dot{x}$ to get

$$\frac{1}{2}\frac{d}{dt}\dot{x}^2 - \omega_o^2 \frac{d}{dt}\cos x = 0$$

- Integrating we find that the energy of the pendulum is conserved

$$\frac{1}{2\omega_o^2}\dot{x}^2 - \cos x = \text{constant} = \text{ energy of the system} = E$$



With $x = \theta$

Non-linear forces

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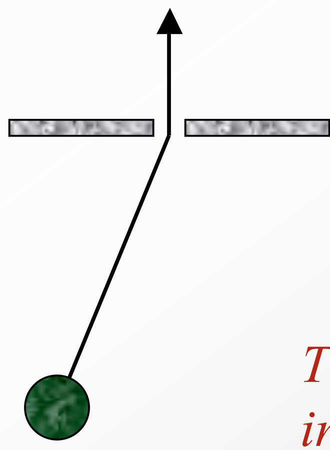
- Beams subject to non-linear forces are commonplace in accelerators
- Examples include
  - Space charge forces in beams with non-uniform charge distributions
  - Forces from magnets high than quadrupoles
  - Electromagnetic interactions of beams with external structures
    - Free Electron Lasers
    - Wakefields

Properties of harmonic oscillators

- Total energy is conserved

$$U = \frac{p^2}{2m} + \frac{m\omega_o^2 x^2}{2}$$

- If there are *slow* changes in m or ω, then $I = U/\omega_o$ *remains invariant*

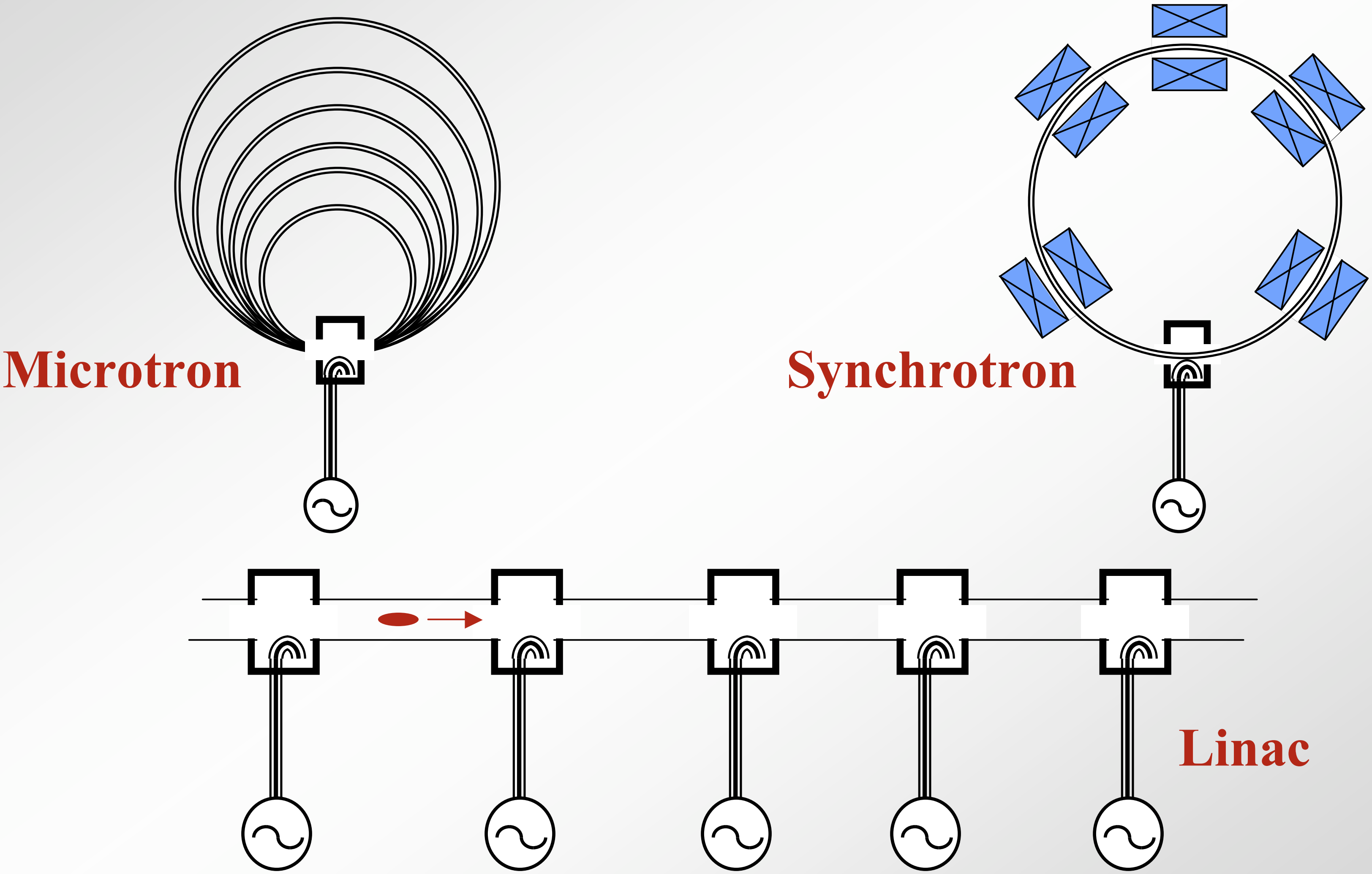


$$\frac{\Delta\omega_o}{\omega_o} = \frac{\Delta U}{U}$$

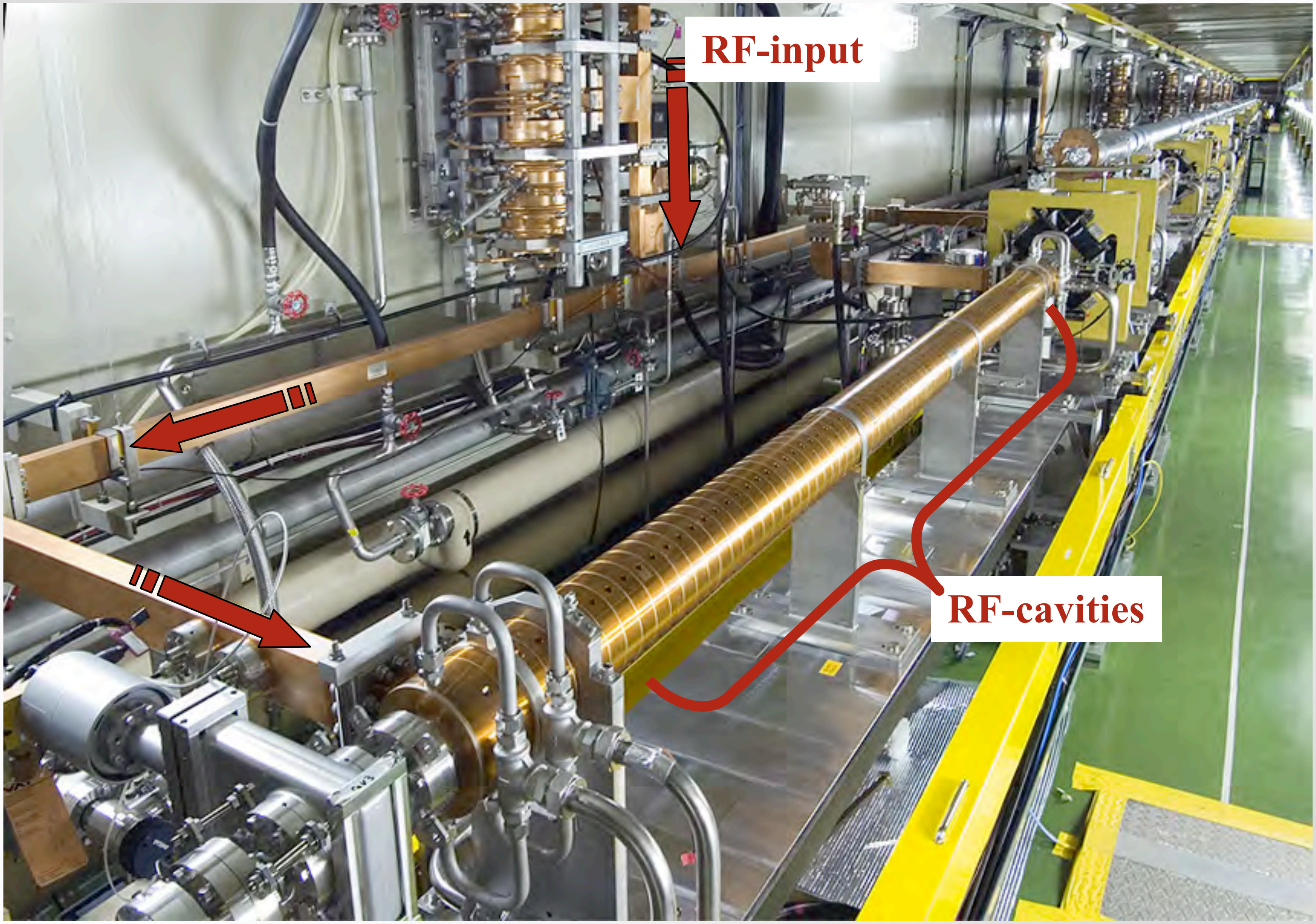
This effect is important as a diagnostic in measuring resonant properties of structures

RF-accelerators & RF-cavities

RF-cativties for acceleration



S-band (~3 GHz) RF linac



Ohm's Law Generalized

- Basic approach is the Fourier analysis of a circuit

- Start with

$$\tilde{V} = Ve^{j(\omega t + \varphi)}$$

- Instead of V = IR where the quantities are real we write

$$\tilde{V}(\omega) = \tilde{I}(\omega)\overleftrightarrow{Z}(\omega)$$

- We know this works for resistors.

$$V(t) = R\ I(t) \Longrightarrow Z_R \text{ is real} = R$$

- What about capacitors & inductors?

Impedance of Capacitors

- For a capacitor

$$I = C\left(\frac{dV}{dt}\right) \Rightarrow \tilde{I} = C\frac{d}{dt}Ve^{j(\omega t+\varphi)} = j\omega C\tilde{V}$$

- So our generalized Ohm's law is

$$\tilde{V} = \tilde{I}\overleftrightarrow{Z}_C$$

where

$$\tilde{Z}_C = \frac{1}{j\omega C}$$

Impedance of Inductors

- For a capacitor

$$V = L\left(\frac{dI}{dt}\right) \Rightarrow \tilde{V} = L\frac{d}{dt}Ie^{j(\omega t+\varphi)} = j\omega L\tilde{I}$$

- So our generalized Ohm's law is

$$\tilde{V} = \tilde{I}\overleftrightarrow{Z}_L$$

Where

$$\tilde{Z}_L = j\omega L$$

Combining impedances

- The algebraic form of Ohm's Law is preserved

==> impedances follow the same rules for combination in series and parallel as for resistors

- For example

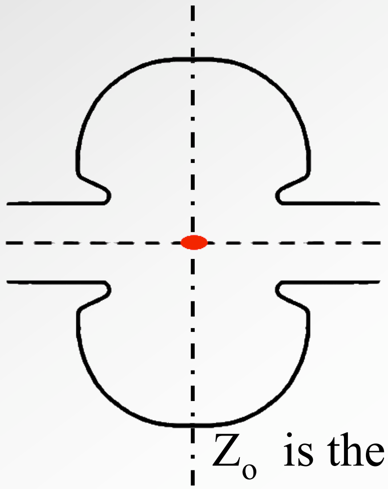
$$\tilde{Z}_s = \tilde{Z}_1 + \tilde{Z}_2$$

$$\tilde{Z}_p = \left[1/\tilde{Z}_1 + 1/\tilde{Z}_2\right]^{-1} = \frac{\tilde{Z}_1 \tilde{Z}_2}{\tilde{Z}_1 + \tilde{Z}_2}$$

- We can now solve circuits using Kirkhoff's laws, *but in the frequency domain*

RF cativties: Basic concepts

- Fields and voltages are complex quantities.
  - For standing wave structures use phasor representation



$$\tilde{V} = V e^{i\omega t} \quad \text{where} \quad V = \left|\tilde{V}\right|$$

At t = 0 particle receives maximum voltage gain

Z_o is the reference plane

- For cavity driven externally, phase of the voltage is

$$\theta = \omega t + \theta_o$$

- For electrons $v \approx c$; therefore $z = z_o + ct$

Basic principles and concepts

- Superposition
- Energy conservation
- Orthogonality (of cavity modes)
- Causality

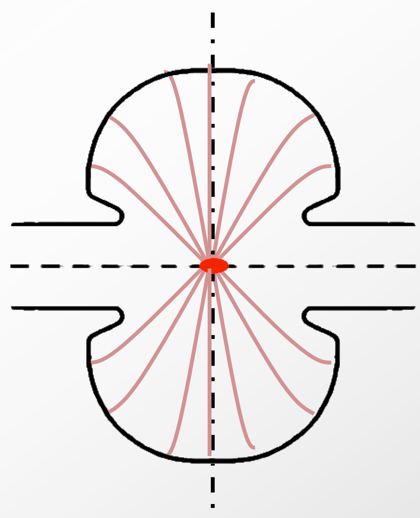
Basic principles:
Reciprocity & superposition

- If you can kick the beam, the beam can kick you

==>

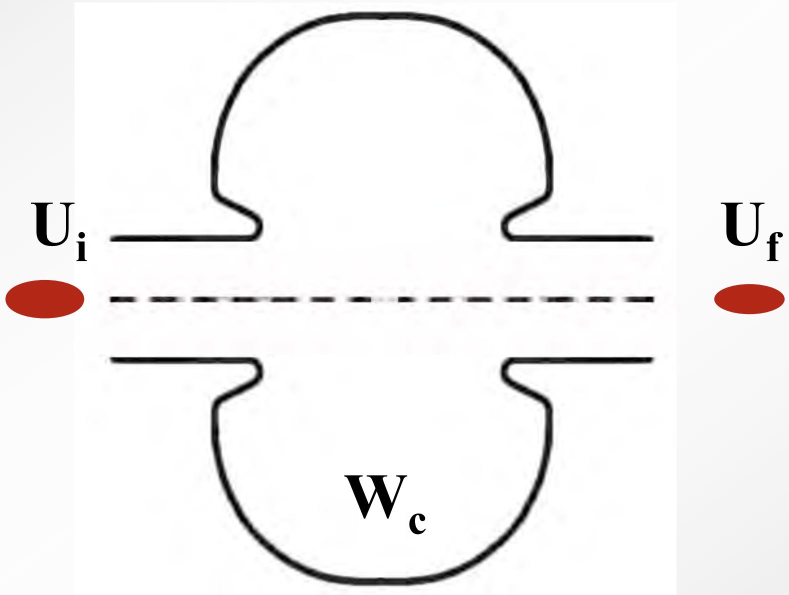
$$\text{Total cavity voltage} = V_{generator} + V_{beam\text{-}induced}$$

$$\text{Fields in cavity} = \mathbf{E}_{generator} + \mathbf{E}_{beam\text{-}induced}$$



Basic principles: Energy conservation

- Total energy in the particles and the cavity is conserved
  - Beam loading



$$\Delta W_c = U_i - U_f$$

Basics: Orthogonality of normal modes

- Each mode in the cavity can be treated independently in computing fields induced by a charge crossing the cavity.
- The total stored energy is equals the sum of the energies in the separate modes.
- The total field is the phasor sum of all the individual mode fields at any instant.

Basic principles: Causality

- There can be no disturbance ahead of a charge moving at the velocity of light.
- In a mode analysis of the growth of the beam-induced field, the field must vanish ahead of the moving charge for each mode.

Basic components of an RF cavity

Outer region: Large, single turn Inductor

Central region: Large plate Capacitor

Power feed from rf - generator